Solution to Assignment 4, MMAT5520

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Exercise 5.2:
1(b). **Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda - 3 & 2 \\ -4 & \lambda + 1 \end{vmatrix} = 0,$$

$$\lambda^2 - 2\lambda + 5 = 0,$$

$$\lambda = 1 \pm 2i,$$

For $\lambda_1 = 1 + 2i$, consider

$$\begin{pmatrix} 2i - 2 & 2 \\ -4 & 2 + 2i \end{pmatrix} v = 0,$$

$v_1 = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}$ is an eigenvector of A corresponding to $1 + 2i$.
For $\lambda_2 = 1 - 2i$, consider

$$\begin{pmatrix} -2i - 2 & 2 \\ -4 & 2 - 2i \end{pmatrix} v = 0,$$

$v_2 = \begin{pmatrix} 1 \\ 1 + i \end{pmatrix}$ is an eigenvector of A corresponding to $1 - 2i$.
Let $Q = \begin{pmatrix} 1 & 1 \\ 1 - i & 1 + i \end{pmatrix}$, then we have $Q^{-1}AQ = \begin{pmatrix} 1 + 2i & 0 \\ 0 & 1 - 2i \end{pmatrix}$.

1(d). **Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ -6 & -11 & \lambda - 6 \end{vmatrix} = 0,$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0,$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0,$$

$$\lambda = 1, 2, 3.$$

For $\lambda_1 = 1$, consider

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -6 & -11 & -5 \end{pmatrix} v = 0,$$

$v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$.

For $\lambda_2 = 2$, consider

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -6 & -11 & -4 \end{pmatrix} v = 0,$$

$v_2 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = 2$.

For $\lambda_3 = 3$, consider

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ -6 & -11 & -3 \end{pmatrix} v = 0,$$

$v_3 = \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_3 = 3$.

Let $Q = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{pmatrix}$, then we have $Q^{-1}AQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

1(e). **Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda - 3 & 2 & 0 \\ 0 & \lambda - 1 & 0 \\ 4 & -4 & \lambda - 1 \end{vmatrix} = 0,$$

$$(\lambda - 1)^2(\lambda - 3) = 0,$$

$$\lambda = 1, 1, 3.$$

For $\lambda_1 = \lambda_2 = 1$, consider

$$\begin{pmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \\ 4 & -4 & 0 \end{pmatrix} v = 0,$$

$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ are two independent eigenvectors corresponding to $\lambda = 1$.

For $\lambda_3 = 3$, consider

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 4 & -4 & 2 \end{pmatrix} v = 0,$$

$v_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 3$.

Let $Q = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$, then we have $Q^{-1}AQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

2. **Soution:** You could find it in "Lecture Notes: Chapter 1-6, Answers to Exercises".

7. **Soution:** You could find it in "Lecture Notes: Chapter 1-6, Answers to Exercises".

10. **Soution:** You could find it in "Lecture Notes: Chapter 1-6, Answers to Exercises".

Exercise 5.3:

1(a).**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda - 5 & 6 \\ -3 & \lambda + 4 \end{vmatrix} = 0,$$

$$(\lambda + 1)(\lambda - 2) = 0,$$

$$\lambda = -1, 2$$

For $\lambda_1 = -1$, consider

$$\begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix} v = 0,$$

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = -1$.
For $\lambda_2 = 2$, consider

$$\begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix} v = 0,$$

$v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = 2$.
Let $Q = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, then we have $Q^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$ and $Q^{-1}AQ = D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$.
Hence

$$\begin{aligned} A^5 &= (QDQ^{-1})^5 = QD^5Q^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}^5 \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 32 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 65 & -66 \\ 33 & -34 \end{pmatrix}. \end{aligned}$$

1(d).**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda - 1 & 5 \\ -1 & \lambda + 1 \end{vmatrix} = 0,$$

$$\lambda^2 + 4 = 0,$$

$$\lambda = \pm 2i$$

For $\lambda_1 = 2i$, consider

$$\begin{pmatrix} -1 + 2i & 5 \\ -1 & 1 + 2i \end{pmatrix} v = 0,$$

$v_1 = \begin{pmatrix} 1 + 2i \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = 2i$.

For $\lambda_2 = -2i$, consider

$$\begin{pmatrix} -1-2i & 5 \\ -1 & 1-2i \end{pmatrix} v = 0,$$

$v_2 = \begin{pmatrix} 1-2i \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = -2i$.
Let $Q = \begin{pmatrix} 1+2i & 1-2i \\ 1 & 1 \end{pmatrix}$, then we have $Q^{-1} = \frac{1}{4i}\begin{pmatrix} 1 & -1+2i \\ -1 & 1+2i \end{pmatrix}$ and $Q^{-1}AQ = D = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$.
Hence

$$\begin{aligned}
A^5 = (QDQ^{-1})^5 &= QD^5Q^{-1} \\
&= \begin{pmatrix} 1+2i & 1-2i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}^5 \frac{1}{4i}\begin{pmatrix} 1 & -1+2i \\ -1 & 1+2i \end{pmatrix} \\
&= \frac{1}{4i}\begin{pmatrix} 1+2i & 1-2i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 32i & 0 \\ 0 & -32i \end{pmatrix}\begin{pmatrix} 1 & -1+2i \\ -1 & 1+2i \end{pmatrix} \\
&= \begin{pmatrix} 16 & -80 \\ 16 & -16 \end{pmatrix}.
\end{aligned}$$

1(e). **Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda-1 & -2 & 1 \\ -2 & \lambda-4 & 2 \\ -3 & -6 & \lambda+3 \end{vmatrix} = 0,$$

$$\lambda^2(\lambda-2) = 0,$$

$$\lambda = 0, 0, 2.$$

For $\lambda_1 = \lambda_2 = 0$, consider

$$\begin{pmatrix} -1 & -2 & 1 \\ -2 & -4 & 2 \\ -3 & -6 & 3 \end{pmatrix} v = 0,$$

$v_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are two independent eigenvectors corresponding to $\lambda = 1$.
For $\lambda_3 = 2$, consider

$$\begin{pmatrix} 1 & -2 & 1 \\ -2 & -2 & 2 \\ -3 & -6 & 5 \end{pmatrix} v = 0,$$

$v_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 2$.
Let $Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$, then we have $Q^{-1} = \frac{1}{2}\begin{pmatrix} -2 & -2 & 2 \\ -3 & -6 & 5 \\ 1 & 2 & -1 \end{pmatrix}$ and $Q^{-1}AQ = D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Hence

$$\begin{aligned} A^5 &= (QDQ^{-1})^5 = QD^5Q^{-1} \\ &= \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}^5 \frac{1}{2} \begin{pmatrix} -2 & -2 & 2 \\ -3 & -6 & 5 \\ 1 & 2 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 32 \end{pmatrix} \begin{pmatrix} -2 & -2 & 2 \\ -3 & -6 & 5 \\ 1 & 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 16 & 32 & -16 \\ 32 & 64 & -32 \\ 48 & 96 & -48 \end{pmatrix}. \end{aligned}$$

Exercise 5.4:

1(a).**Soution:** The characteristic polynomial of A is:

$$\det(xI - A) = \begin{vmatrix} x-5 & 4 \\ -3 & x+2 \end{vmatrix} = (x-1)(x-2).$$

The minimal polynomial of A is $m(x) = (x-1)(x-2) = x^2 - 3x + 2$.

$$A^2 - 3A + 2I = 0, \quad A^2 = 3A - 2I,$$

$$A^3 = 3A^2 - 2A = 3(3A - 2I) - 2A = 7A - 6I,$$
$$A^4 = 7A^2 - 6A = 7(3A - 2I) - 6A = 15A - 14I.$$

$$A^2 - 3A + 2I = 0, \quad A - 3I + 2A^{-1} = 0,$$
$$A^{-1} = -\frac{1}{2}A + \frac{3}{2}I.$$

1(b).**Soution:** The characteristic polynomial of A is:

$$\det(xI - A) = \begin{vmatrix} x-3 & 2 \\ -2 & x+1 \end{vmatrix} = (x-1)^2.$$

The minimal polynomial of A is either $x-1$ or $(x-1)^2$.
Since $A - I = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \neq \mathbf{0}$, the minimal polynomial of A is $m(x) = (x-1)^2 = x^2 - 2x + 1$.

$$A^2 - 2A + I = 0, \quad A^2 = 2A - I,$$

$$A^3 = 2A^2 - A = 2(2A - I) - A = 3A - 2I,$$
$$A^4 = 3A^2 - 2A = 3(2A - I) - 2A = 4A - 3I.$$

$$A^2 - 2A + I = 0, \quad A - 2I + A^{-1} = 0,$$
$$A^{-1} = -A + 2I.$$

1(d).**Soution:** The characteristic polynomial of A is:

$$\det(xI-A) = \begin{vmatrix} x+1 & -1 & 0 \\ 4 & x-3 & 0 \\ -1 & 0 & x-2 \end{vmatrix} = (x-1)^2(x-2).$$

The minimal polynomial of A is either $(x-1)(x-2)$ or $(x-1)^2(x-2)$.
Since $(A-I)(A-2I) = \begin{pmatrix} -3 & 1 & 0 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 4 & -2 & 0 \\ -2 & 1 & 0 \end{pmatrix} \neq \mathbf{0}$, the minimal polynomial of A is $m(x) = (x-1)^2(x-2) = x^3 - 4x^2 + 5x - 2$.

$$A^3 - 4A^2 + 5A - 2I = 0,$$

$$A^3 = 4A^2 - 5A + 2I,$$

$$A^4 = 4A^3 - 5A^2 + 2A = 4(4A^2 - 5A + 2I) - 5A^2 + 2A = 11A^2 - 18A + 8I.$$

$$A^3 - 4A^2 + 5A - 2I = 0, \quad A^2 - 4A + 5I - 2A^{-1} = 0,$$
$$A^{-1} = \frac{1}{2}A^2 - 2A + \frac{5}{2}I.$$

1(e).**Soution:** The characteristic polynomial of A is:

$$\det(xI-A) = \begin{vmatrix} x-3 & -1 & -1 \\ -2 & x-4 & -2 \\ 1 & 1 & x-1 \end{vmatrix} = (x-2)^2(x-4).$$

The minimal polynomial of A is either $(x-2)(x-4)$ or $(x-2)^2(x-4)$.
Since $(A-2I)(A-4I) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, the minimal polynomial of A is $m(x) = (x-2)(x-4) = x^2 - 6x + 8$.

$$A^2 - 6A + 8I = 0, \quad A^2 = 6A - 8I,$$

$$A^3 = 6A^2 - 8A = 6(6A - 8I) - 8A = 28A - 48I,$$
$$A^4 = 28A^2 - 48A = 28(6A - 8I) - 48A = 120A - 224I.$$

$$A^2 - 6A + 8I = 0, \quad A - 6I + 8A^{-1} = 0,$$
$$A^{-1} = -\frac{1}{8}A + \frac{3}{4}I.$$